

Notes on hyperelliptic fibrations of genus 3, I

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Abstract

We shall study the structure of hyperelliptic fibrations of genus 3, from the view point given by Catanese and Pignatelli in [9]. Here by a hyperelliptic fibration of genus 3, we mean a connected surjective morphism $f : S \rightarrow B$ from a nonsingular complex algebraic surface S to a nonsingular complex projective curve B with general fiber hyperelliptic of genus 3. In this part I, we shall give a structure theorem for such fibrations for the case of $f : S \rightarrow B$ with all fibers 2-connected. The resulting structure theorem is similar to one given in [9] for genus 2 fibrations. We shall also give, for the case of B projective line, sufficient conditions for the existence of such fibrations $f : S \rightarrow B$'s from the view point of our structure theorem, prove the uniqueness of the deformation type and the simply connectedness of S for some cases, and give some examples including those with simply connected S and slope 3.6 and those with minimal regular S with geometric genus $p_g = 4$ and the first Chern number $c_1^2 = 8$. The last example turns out to be a member of the family \mathcal{M}_0 given in Bauer–Pignatelli [4].

1 Introduction

As is known to every algebro-geometers, the study of a fibration $S \rightarrow B$ of a surface S over a nonsingular curve B has been an important branch of algebraic geometry, even from early years of Italian school ([2], [12]). This is not only because a fibration itself is of an object of interest, but also because the study of a fibration of a surface gives important information on the structure of the surface itself. In fact one can easily recall the important roles that the elliptic fibrations and ruled structures play in Enriques–Kodaira classification of algebraic and analytic surfaces. After Kodaira's works on

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elliptic fibrations [17], [18], there are many works on fibrations, for which several approaches have been developed (e.g., [25], [24], [16], [29], [22], [6]).

One of the modern approaches for the study of fibrations of surfaces is that through relative canonical algebras. In this approach, the first step is the study of the canonical algebras of the fibers (e.g., those done for the case of the genus of the fibers ≤ 3 by Mendes Lopes in her thesis [22]), since by the Krull–Azumaya lemma, the study of the local structures of the relative canonical algebras can be to some extent reduced to it. As one can see from for example [27], [20], and [19] such study is important and useful for the study of global structures of the surfaces. We notice that today there are several attempts and deep results on the local structures of the relative canonical algebras and their application to the study of global structures of the fibrations (see [1]), and that as for the global structures of relative canonical algebras, much less are known compared to the case of the local structures, although we have some general theorems, e.g., Fujita’s results on semi-positivity of the direct images of relative canonical sheaves ([13]), as our basic ingredients for the study in this direction.

Recently, in [9], F. Catanese and R. Pignatelli successfully developed a new method for the case of genus 2 fibrations and (2-connected) genus 3 non-hyperelliptic fibrations. They introduced the notion of admissible 5-tuple, which is a collection of data extracted from the structure of the relative canonical algebra, and showed that the 5-tuple completely determines the global structure of the relative canonical model $X \rightarrow B$ of the fibration $S \rightarrow B$. This method turned out to be powerful. In fact, in the same paper, using this result, they were able to give a half-page proof of Bombieri’s result on bicanonical pencils of numerical Godeaux surfaces, and also to prove that the moduli space of minimal surfaces with $c_1^2 = 3$, $p_g = q = 1$, and with albanese fibers of genus 2 has exactly three connected components (all irreducible), thus completing the classification of surfaces with $c_1^2 = 3$ and $p_g = q = 1$ initiated in [6] and [7] (here as usual, c_1^2 , p_g , and q denote the first Chern number, the geometric genus, and the irregularity, respectively, of a surface).

In the present paper, we study the next steps, and establish a theorem (similar to those in [9]) for the easiest case, i.e., the case of hyperelliptic fibrations $S \rightarrow B$ of genus 3 with all fibers 2-connected (Theorem 1). The resulting theorem is similar to those by Catanese and Pignatelli, and is the existence and description of one-to-one correspondence between the isomorphism classes of fibrations $S \rightarrow B$ as above and the isomorphism classes of admissible 5-tuples, which will be introduced in Section 2. We shall also give, for the case of base curve B projective line, sufficient conditions for the existence of our fibrations $f : S \rightarrow B$ ’s from the view point of 5-tuples

(Proposition 6), show for some cases the uniqueness of the deformation type and the simply connectedness of S (Theorem 2, Proposition 7), and give some examples including those with topologically simply connected S and slope 3.6 (Remark 6) and those with minimal regular S with $p_g = 4$ and $c_1^2 = 8$ (Proposition 8). This last example turns out to belong to the family \mathcal{M}_0 defined in Bauer–Pignatelli [4], where they classified minimal regular surfaces with $p_g = 4$ and $c_1^2 = 8$ with canonical involution. The last example is of our special interest, since the study of surfaces with $p_g = 4$ has long history from Enriques [12] (see also [10]), and after the complete classification of the case $c_1^2 = 7$ by I. Bauer [3], the next object is the case $c_1^2 = 8$ (see [10], [4], [8], [26], [28]).

Although the 2-connectedness condition on the fibers is strong, our main result (which covers only the simplest case) is already useful to produce some interesting examples. In fact, as the first step for this purpose, the structure theorem for non-hyperelliptic deformations of the genus 3 hyperelliptic fibrations $S \rightarrow B$ as above will be given in the part II of this series, and there the above mentioned surfaces with $p_g = 4$ and $c_1^2 = 8$ will be deformed to a family of surfaces with non-hyperelliptic genus 3 fibrations. Here one might also notice that one of the advantages of treating our hyperelliptic fibrations through relative canonical algebras (in stead of through double cover descriptions) lies in that we can easily connect them to non-hyperelliptic deformations, as can be seen also from the results in [22].

The present paper is organized as follows. In Section 2, we shall introduce the notion of 5-tuple and state the main theorem. Propositions 1 and 2 explain how the isomorphism classes of fibrations and those of 5-tuples correspond. In Section 3, we shall study the global structure of the relative canonical algebras of our fibrations $S \rightarrow B$. The key results are Propositions 3 and 4, which describe the structure of relative canonical algebras of our $S \rightarrow B$ (see also Remark 2). In Section 4, using the computation in Section 3, we shall prove our main theorem, i.e., Theorem 1. Finally, in Section 5, we shall give sufficient conditions for the existence of fibrations and study some examples. While Proposition 6 gives sufficient conditions for the existence of admissible 5-tuples, Theorem 2, Propositions 7 and 8, and Remarks 5 and 6, etc., study examples.

Throughout this paper, we work over the complex number field \mathbb{C} .

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NOTATION

In this article, the symbol k always denotes the complex number field \mathbb{C} . If V is a locally free sheaf of finite rank on a scheme, $\mathrm{rk} V$ denotes its rank. The symbols $\mathcal{S}(V)$ and $\mathrm{Sym}^n V$ denote the symmetric tensor algebra (of V) and its homogeneous part of degree n , respectively. If R is a graded algebra, R_j denotes its homogeneous part of degree j . Thus, for example, for the polynomial ring $k[x_0, x_1, x_2]$ over the complex number field $k = \mathbb{C}$, an element in $k[x_0, x_1, x_2]_j$ is a homogeneous polynomial in x_0, x_1, x_2 of degree j . The symbol \amalg means taking the disjoint union of sets. If p is a point of a scheme, $k(p)$ denotes the residue field at p of this scheme. If \mathcal{F} is a sheaf on a scheme, \mathcal{F}_p denotes the stalk at p of \mathcal{F} . If moreover the scheme is over k , and \mathcal{F} is coherent, $h^i(\mathcal{F})$ denotes the dimension over k of the i -th cohomology group $H^i(\mathcal{F})$ of \mathcal{F} . If S is a scheme, and D , a Cartier divisor on S , then \mathcal{O}_S and $\mathcal{O}_S(D)$ denote the structure sheaf of S and the invertible sheaf associated to D , respectively. If S is projective and smooth over k , the symbol K_S as usual denotes the canonical divisor of S .

2 Statement of the main result

In this section, we shall state the main theorem. The result is the existence of one-to-one correspondence between isomorphism classes of a kind of data, which we shall call 5-tuples, and isomorphism classes of genus 3 hyperelliptic fibrations with all fibers 2-connected. In order to state the result, we first introduce the notion of 5-tuple, and observe how to associate a fibration to a 5-tuple, and then a 5-tuple to a fibration. And then we state the main theorem.

2.1 From 5-tuples to fibrations

First, let us define the 5-tuple and observe how to associate a genus 3 fibration to it. Let B be a smooth projective curve over the complex number field $k = \mathbb{C}$. Let V_1 and V_2^+ be locally free sheaves on B of rank $\mathrm{rk} V_1 = 3$ and $\mathrm{rk} V_2^+ = 5$, respectively.

Assume that we are given a surjective morphism $\sigma_2 : \mathrm{Sym}^2 V_1 \rightarrow V_2^+$ of sheaves. Then the kernel $L = \ker \sigma_2$ of σ_2 is invertible, and for each natural number $n \in \mathbb{N}$ the natural inclusion $L \rightarrow \mathrm{Sym}^2 V_1$ induces an injective morphism $L \otimes \mathrm{Sym}^{n-2} V_1 \rightarrow \mathrm{Sym}^n V_1$. So we define the coherent \mathcal{O}_B -module

\mathcal{A}_n by the following short exact sequence:

$$0 \rightarrow L \otimes \mathrm{Sym}^{n-2} V_1 \rightarrow \mathrm{Sym}^n V_1 \rightarrow \mathcal{A}_n \rightarrow 0.$$

Then \mathcal{A}_n is a locally free sheaf of rank $2n + 1$. Let $\mathcal{S}(V_1)$ be the symmetric \mathcal{O}_B -algebra associated to the \mathcal{O}_B -module V_1 , and put $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$. Then, via the natural projection $\mathcal{S}(V_1) \rightarrow \mathcal{A}$, the algebra structure of $\mathcal{S}(V_1)$ induces a quasi-coherent graded \mathcal{O}_B -algebra structure on the direct sum $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n$.

Using these graded algebras, we define the two varieties \mathcal{C} and \mathbb{P} by $\mathcal{C} = \mathrm{Proj} \mathcal{A}$ and $\mathbb{P} = \mathrm{Proj} \mathcal{S}(V_1) = \mathbb{P}(V_1)$. By the projection $\mathcal{S}(V_1) \rightarrow \mathcal{A}$, we obtain a natural closed embedding $\mathcal{C} \rightarrow \mathbb{P}$ over the curve B .

Set $V_2^- = (\det V_1) \otimes L^{\otimes(-1)}$ and $\mathcal{R} = \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$, where $\mathcal{A}[-2]$ is the (-2) -shift of the graded algebra \mathcal{A} . Then the \mathcal{O}_B -module \mathcal{R} allows a natural graded \mathcal{A} -module structure. Assume moreover that we are given an element

$$\begin{aligned} \delta \in \mathrm{Hom}_{\mathcal{O}_B}((V_2^-)^{\otimes 2}, \mathcal{A}_4) &\simeq H^0(B, \mathcal{A}_4 \otimes (V_2^-)^{\otimes(-2)}) \\ &\simeq H^0(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(4) \otimes \pi_{\mathcal{C}}^*(V_2^-)^{\otimes(-2)}), \end{aligned}$$

where $\pi_{\mathcal{C}} : \mathcal{C} \rightarrow B$ is the natural projection.

Since $\delta : (V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ induces a natural morphism of \mathcal{O}_B -modules $(\mathcal{A}[-2] \otimes V_2^-)_m \otimes (\mathcal{A}[-2] \otimes V_2^-)_n \rightarrow \mathcal{A}_{m+n}$ for each $m, n \geq 2$, and since the graded \mathcal{A} -module structure on \mathcal{R} gives $\mathcal{R}_m \otimes \mathcal{A}_n \rightarrow \mathcal{R}_{m+n}$, the element δ determines a graded \mathcal{O}_B -algebra structure on the \mathcal{A} -module $\mathcal{R} = \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$. Note that the natural inclusion $\mathcal{A} \rightarrow \mathcal{R} = \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$ of the first direct summand \mathcal{A} is a morphism of graded \mathcal{O}_B -algebras. Thus if we put $X = \mathrm{Proj} \mathcal{R}$, we obtain the following commutative diagram:

$$\begin{array}{ccccc} X = \mathrm{Proj} \mathcal{R} & \longrightarrow & \mathcal{C} = \mathrm{Proj} \mathcal{A} & \longrightarrow & \mathbb{P} = \mathrm{Proj} \mathcal{S}(V_1) \\ \downarrow \bar{f} & & \downarrow \pi_{\mathcal{C}} & & \downarrow \pi_{\mathbb{P}} \\ B & \xrightarrow{=} & B & \xrightarrow{=} & B, \end{array}$$

where \bar{f} , $\pi_{\mathcal{C}}$, and $\pi_{\mathbb{P}}$ are the natural projections.

We shall call $(B, V_1, V_2^+, \sigma_2, \delta)$ as above a 5-tuple for relatively minimal (2-connected) hyperelliptic fibrations of genus 3, or simply, a 5-tuple. We shall call $\bar{f} : X \rightarrow B$ as above the relative canonical model associated to the 5-tuple, or simply, the associated relative canonical model. If the associated relative canonical model $\bar{f} : X \rightarrow B$ satisfies the two conditions

- I) \mathcal{C} has at most rational double points as its singularities, and
- II) X has at most rational double points as its singularities,

we say that the 5-tuple $(B, V_1, V_2^+, \sigma_2, \delta)$ is admissible.

We shall prove the following proposition in Section 4.

Proposition 1. *Let $(B, V_1, V_2^+, \sigma_2, \delta)$ be an admissible 5-tuple, $\bar{f} : X \rightarrow B$, its associated relative canonical model, and $S \rightarrow X$, the minimal resolution of singularities of X . Denote by $f : S \rightarrow B$ the composite of the two morphisms $S \rightarrow X$ and $\bar{f} : X \rightarrow B$. Then $f : S \rightarrow B$ is a relatively minimal hyperelliptic fibration of genus 3 all of whose fibers are 2-connected.*

2.2 From fibrations to 5-tuples

Next, let us observe how to associate a 5-tuple to a hyperelliptic fibration of genus 3. Let $f : S \rightarrow B$ be a relatively minimal hyperelliptic fibration of genus 3 all of whose fibers are 2-connected. Let $\omega_{S|B}$ be the relative dualizing sheaf of f , and $V_n = f_*(\omega_{S|B}^{\otimes n})$, the direct image sheaf of $\omega_{S|B}^{\otimes n}$ by f . Then we have a natural decomposition $V_n = V_n^+ \oplus V_n^-$ into eigen-sheaves: V_n^+ and V_n^- are eigen-sheaves of eigenvalue $+1$ and -1 , respectively, with respect to the action by the hyperelliptic involution of f . It is easy to see that $\text{rk } V_1 (= V_1^-) = 3$ and $\text{rk } V_2^+ = 5$. Now let $\sigma_2 : \text{Sym}^2 V_1 \rightarrow V_2^+$ be the natural morphism induced by the multiplication structure of the relative canonical algebra $\mathcal{R} = \bigoplus_{n=0}^{\infty} V_n$ of f , and $L = \ker \sigma_2$, its kernel. We denote by $\delta : (V_2^-)^{\otimes 2} \rightarrow V_4^+$ the natural morphism induced by the multiplication structure of \mathcal{R} .

We shall prove the following proposition in Section 4.

Proposition 2. *Let $f : S \rightarrow B$ a relatively minimal hyperelliptic fibration of genus 3 all of whose fibers are 2-connected, $\mathcal{R} = \bigoplus_{n=0}^{\infty} V_n$, its relative canonical algebra, and $\mathcal{A} \subset \mathcal{R}$, the graded \mathcal{O}_B -subalgebra generated by the degree 1 part $V_1 = V_1^-$ in \mathcal{R} . Then the degree 4 part \mathcal{A}_4 of \mathcal{A} coincides with V_4^+ , and $(B, V_1, V_2^+, \sigma_2, \delta)$ above forms an admissible 5-tuple for relatively minimal hyperelliptic fibrations of genus 3.*

Given a fibration $f : S \rightarrow B$ as above, we shall call $(B, V_1, V_2^+, \sigma_2, \delta)$ as in Proposition 2 the 5-tuple associated to f .

2.3 Main theorem

Under the terminology as above, our main theorem, which we shall prove in Section 4, is the following:

Theorem 1. *Let B be a smooth projective curve over a complex number field \mathbb{C} . Then via the associations given in Propositions 1 and 2, which are mutually inverse, the isomorphism classes of relatively minimal genus 3 hyperelliptic fibrations with all fibers 2-connected are in one-to-one correspondence with the isomorphism classes of admissible 5-tuples $(B, V_1, V_2^+, \sigma_2, \delta)$'s.*

Moreover, given an admissible 5-tuple $(B, V_1, V_2^+, \sigma_2, \delta)$, the resulting surface S appearing in the associated fibration $f : S \rightarrow B$ has numerical invariants

$$\begin{aligned}\chi(\mathcal{O}_S) &= \deg V_1 + 2(b - 1) \\ c_1^2(S) &= 4 \deg V_1 - 2 \deg L + 16(b - 1)\end{aligned}$$

where L is the kernel of the morphism σ_2 , and $b = g(B)$, the genus of the base curve B .

3 The structure of relative canonical algebra

In this section, we shall study the structure of the relative canonical algebras for our fibrations. Let $f : S \rightarrow B$ be a relatively minimal hyperelliptic genus 3 fibration with all fibers 2-connected. Recall that we have a natural decomposition $V_n = V_n^+ \oplus V_n^-$ of the direct image sheaf $V_n = f_*(\omega_{S|B}^{\otimes n})$ induced by the action on V_n by the hyperelliptic involution of f . It is straightforward to see that both V_n^+ and V_n^- are locally free sheaves and to see that we have

$$\begin{aligned}\mathrm{rk} V_n^\pm &= 2n + 1 \\ \mathrm{rk} V_n^\mp &= 2n - 3,\end{aligned}$$

where the symbol \pm stands for $+$ if n is even, for $-$ if n is odd, and the symbol \mp stands for $-$ if n is even, for $+$ if n is odd. In using the symbols \pm and \mp , we shall keep this rule throughout this paper.

Note that we have in particular

$$\mathrm{rk} V_1 = \mathrm{rk} V_1^- = 3, \quad \mathrm{rk} V_2^+ = 5, \quad \mathrm{rk} V_2^- = 1.$$

Lemma 3.1. *Let F be a fiber of the fibration f , and $R(F, K_F)$, the canonical ring of the fiber F . Then*

$$R(F, K_F) \simeq k[x_0, x_1, x_2, y]/(Q, y^2 - P)$$

as graded $k = \mathbb{C}$ -algebras, where $\deg x_i = 1$ for $0 \leq i \leq 2$, $\deg y = 2$, and

$$\begin{aligned}Q &= x_2^2 - Q_1(x_0, x_1)x_2 - Q_2(x_0, x_1) \in k[x_0, x_1, x_2]_2 \\ P &= P_3(x_0, x_1)x_2 + P_4(x_0, x_1) \in k[x_0, x_1, x_2]_4.\end{aligned}$$

Here $Q_j(x_0, x_1) \in k[x_0, x_1]_j$ ($j = 1, 2$) and $P_i(x_0, x_1) \in k[x_0, x_1]_i$ ($i = 3, 4$) are homogeneous polynomials in x_0, x_1 of degree j and i , respectively.

Proof. See Mendes Lopes [22, Theorem 6.1, p.198]. Note that in our case the fiber F is hyperelliptic. \square

Remark 1. Note that in Lemma 3.1 the element y is a base of 1-dimensional linear space $R(F, K_F)_2^-$, i.e., that of the eigenspace of the eigenvalue -1 with respect to the action by the hyperelliptic involution on the homogeneous part $R(F, K_F)_2$ of degree 2. The action by the hyperelliptic involution on $R(F, K_F)$ is given by

$$(x_0, x_1, x_2, y) \mapsto (-x_0, -x_1, -x_2, -y).$$

From Lemma 3.1 we infer the following:

Lemma 3.2. *Let $(*)_n$ and $(**)_n$ be the two sets of monomials of weighted degree n in x_0, x_1, x_2 , and y defined as follows:*

$$\begin{aligned} (*)_n &= \{(\text{monomials in } x_0, x_1 \text{ of degree } n)\} \\ &\quad \amalg \{(\text{monomials in } x_0, x_1 \text{ of degree } n-1) \times x_2\} \\ (**) _n &= \{(\text{monomials in } x_0, x_1 \text{ of degree } n-2) \times y\} \\ &\quad \amalg \{(\text{monomials in } x_0, x_1 \text{ of degree } n-3) \times x_2 y\}. \end{aligned}$$

Then for any integer $n \geq 2$, the set $(*)_n$ forms a base of $R(F, K_F)_n^\pm$, and the set $(**) _n$, a base of $R(F, K_F)_n^\mp$, where $R(F, K_F)_n^+$ and $R(F, K_F)_n^-$ are eigenspaces of eigenvalues 1 and -1 , respectively, with respect to the action by the hyperelliptic involution on the homogeneous part $R(F, K_F)_n$ of degree n . Here as before the symbol \pm stands for $+$ if n is even, for $-$ if n is odd, and the symbol \mp stands for $-$ if n is even, for $+$ if n is odd.

Proof. Note that $y^2 - P$ is monic in y and that Q is monic in x_2 . Thus the assertion follows immediately from Lemma 3.1. \square

Now let us denote by σ_n the natural morphism $\sigma_n : \text{Sym}^n V_1 \rightarrow V_n^\pm$ determined by the multiplication structure of the relative canonical algebra $\mathcal{R}(f) = \bigoplus_{n=0}^\infty V_n$. Let us moreover denote by $L = \ker \sigma_2$ the kernel of the morphism $\sigma_2 : \text{Sym}^2 V_1 \rightarrow V_2^+$. Then since $\sigma_2 \otimes k(p) : (\text{Sym}^2 V_1) \otimes k(p) \rightarrow V_2^+ \otimes k(p)$ is surjective for any point $p = f(F)$ by Lemma 3.2, we obtain by the Krull–Azumaya Lemma (i.e., Nakayama’s Lemma) the exactness of the complex

$$0 \rightarrow L \rightarrow \text{Sym}^2 V_1 \rightarrow V_2^+ \rightarrow 0.$$

Since we have $\text{rk Sym}^2 V_1 = 6$ and $\text{rk } V_2^+ = 5$, we see that the sheaf L is invertible on B .

The next two lemmas follow immediately from the Krull–Azumaya Lemma:

Lemma 3.3. *Let F be a (closed) fiber of f , and P and Q , polynomials as in Lemma 3.1 in x_0, x_1 , and x_2 . Then there exists a neighbourhood U of the point $p = f(F) \in B$, such that the relations Q and $y^2 - P$ in $R(F, K_F)$ lift to the relations \tilde{Q} and $y^2 - \tilde{P}$, respectively, of the form*

$$\begin{aligned}\tilde{Q} &= x_2^2 - \tilde{Q}_1(x_0, x_1)x_2 - \tilde{Q}_2(x_0, x_1) \\ y^2 - \tilde{P} &= y^2 - \tilde{P}_3(x_0, x_1)x_2 - \tilde{P}_4(x_0, x_1)\end{aligned}$$

in the relative canonical algebra $\mathcal{R} = \mathcal{R}(f)$. Here $\tilde{Q}_j(x_0, x_1) \in \mathcal{O}_B(U)[x_0, x_1]_j$ ($j = 1, 2$) and $\tilde{P}_i(x_0, x_1) \in \mathcal{O}_B(U)[x_0, x_1]_i$ ($i = 3, 4$) are homogeneous polynomials with coefficients in $\mathcal{O}_B(U)$ of degree j and i , respectively, and satisfy $\tilde{Q}_j|_{f(F)} = Q_j$ and $\tilde{P}_i|_{f(F)} = P_i$.

Lemma 3.4. *Let $f : S \rightarrow B$ be a fibration as in the beginning of this section. Then for any integer $n \geq 2$, the morphism $\sigma_n : \text{Sym}^n V_1 \rightarrow V_n^\pm$ is surjective.*

In fact, take liftings to the stalk $V_{1,p}$ at $p = f(F)$ of the elements x_0, x_1 , and $x_2 \in V_{1,p} \otimes k(p)$ in Lemma 3.1. We use the same symbols x_0, x_1 , and x_2 for the respective liftings to the stalk $V_{1,p}$. Then the polynomial Q in Lemma 3.1 defines an element of the stalk $V_{2,p}^+$, for which we use the same symbol Q . By the Krull–Azumaya Lemma, the stalk $V_{2,p}^+$ at p of the sheaf V_2^+ is generated by the elements in $(*)_2$, and since $Q \mapsto 0$ by the natural projection $V_{2,p}^+ \rightarrow V_{2,p}^+ \otimes k(p)$, we see that the polynomial Q is in $V_{2,p}^+$ a linear combination of elements in the set $(*)_2$ with coefficients in \mathfrak{M}_p , where \mathfrak{M}_p is the maximal ideal of the local ring $\mathcal{O}_{B,p}$ at p . This shows the existence of \tilde{Q} in Lemma 3.3. The proof for the existence of \tilde{P} in Lemma 3.3 is the same. Lemma 3.4 can be proved by the same argument as that for the surjectivity of σ_2 .

Now let us define the graded \mathcal{O}_B -algebra $\mathcal{A} = \bigoplus_{n=0}^\infty \mathcal{A}_n$ as the \mathcal{O}_B -subalgebra generated by $V_1 = V_1^-$ in the relative canonical algebra $\mathcal{R} = \mathcal{R}(f) = \bigoplus_{n=0}^\infty V_n$. By Lemma 3.4, we see that

$$\mathcal{A}_n = V_n^\pm.$$

The next lemma says that the graded \mathcal{O}_B -algebra structure of \mathcal{A} is completely determined by the natural inclusion $L \rightarrow \text{Sym}^2 V_1$ (and hence by the projection $\sigma_2 : \text{Sym}^2 V_1 \rightarrow V_2^+$).

Lemma 3.5. *Consider the natural morphism*

$$L \otimes \text{Sym}^{n-2} V_1 \rightarrow \text{Sym}^n V_1 : l \otimes q \mapsto \tilde{Q}q$$

induced by the inclusion $L \rightarrow \mathrm{Sym}^2 V_1 : l \mapsto \tilde{Q}$, where q is a local section to $\mathrm{Sym}^{n-2} V_1$, and l , the local base of L corresponding to the local section \tilde{Q} to $\mathrm{Sym}^2 V_1$ as in Lemma 3.3. Then the natural complex

$$0 \rightarrow L \otimes \mathrm{Sym}^{n-2} V_1 \rightarrow \mathrm{Sym}^n V_1 \rightarrow \mathcal{A}_n \rightarrow 0 \quad (1)$$

is exact. Moreover for any closed point $p \in B$, the complex (1) tensored by the residue field $k(p)$ is also exact.

Proof. Since we have $\tilde{Q}|_p = Q = x_2^2 - x_2 Q_1(x_0, x_1) - Q_2(x_0, x_1)$, the morphism $(L \otimes \mathrm{Sym}^{n-2} V_1) \otimes k(p) \rightarrow (\mathrm{Sym}^n V_1) \otimes k(p)$ is injective and has maximal rank. But we have

$$\begin{aligned} \mathrm{rk}(\mathrm{Sym}^n V_1) &= (n+2)(n+1)/2, \\ \mathrm{rk}(L \otimes \mathrm{Sym}^{n-2} V_1) &= n(n-1)/2, \\ \mathrm{rk} \mathcal{A}_n &= \mathrm{rk} V n^\pm = 2n+1, \end{aligned}$$

and hence $\mathrm{rk}(\mathrm{Sym}^n V_1) - \mathrm{rk}(L \otimes \mathrm{Sym}^{n-2} V_1) = \mathrm{rk} \mathcal{A}_n$. Thus the complex (1) tensored by $k(p)$ is exact at every closed point $p \in B$. This implies in particular the surjectivity of $L \otimes \mathrm{Sym}^{n-2} V_1 \rightarrow \ker \sigma_n$. Since two locally free sheaves $L \otimes \mathrm{Sym}^{n-2} V_1$ and $\ker \sigma_n$ have the same rank $n(n-1)/2$, this shows that $L \otimes \mathrm{Sym}^{n-2} V_1 \rightarrow \ker \sigma_n$ is an isomorphism, hence the assertion. \square

Lemma 3.6. *The morphism*

$$\mathcal{A}_{n-2} \otimes V_2^- \rightarrow V_n^\mp : q \otimes y \mapsto qy \quad (2)$$

is an isomorphism.

Proof. By Lemma 3.2, the morphism $(\mathcal{A}_{n-2} \otimes V_2^-) \otimes k(p) \rightarrow V_n^\mp \otimes k(p)$ is an isomorphism for any closed $p \in B$. Thus by the Krull–Azumaya Lemma, $\mathcal{A}_{n-2} \otimes V_2^- \rightarrow V_n^\mp$ is surjective. But both hands of this morphism are locally free sheaves of the same rank $2n-3$, hence the assertion. \square

Thus we obtain the following proposition:

Proposition 3. *Let $\mathcal{A} \subset \mathcal{R}$ be the graded subalgebra as above. Then*

$$\mathcal{R} \simeq \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$$

as graded \mathcal{A} -modules.

We denote by $\delta \in \mathrm{Hom}_{\mathcal{O}_B}((V_2^-)^{\otimes 2}, \mathcal{A}_4)$ the \mathcal{O}_B -module homomorphism from $(V_2^-)^{\otimes 2}$ to $\mathcal{A}_4 = V_4^+$ determined by the multiplication structure of the relative canonical algebra \mathcal{R} (notice that $V_2^- \subset \mathcal{R}_2$ and $\mathcal{A}_4 = V_4^+ \subset \mathcal{R}_4$).

Remark 2. By Remark 1 and Lemma 3.3, the section y is a local base of the invertible sheaf V_2^- . Note that with this y , the morphism $\delta : (V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ is given by $y^2 \mapsto \tilde{P} = \tilde{P}_3(x_0, x_1)x_2 + \tilde{P}_4(x_0, x_1)$ where \tilde{P} , \tilde{P}_3 , and \tilde{P}_4 are polynomials as in Lemma 3.3. Moreover, via the isomorphism (2) in Lemma 3.6, the multiplication morphism $V_m^\mp \otimes V_n^\mp \rightarrow V_{m+n}^\pm = \mathcal{A}_{m+n}$ is given by

$$(\mathcal{A}_{m-2} \otimes V_2^-) \otimes (\mathcal{A}_{n-2} \otimes V_2^-) \rightarrow \mathcal{A}_{m+n} : (\alpha \otimes y) \otimes (\beta \otimes y) \mapsto (\alpha\beta)\delta(y^2).$$

From this together with the short exact sequence (1) in Lemma 3.5, we see that the \mathcal{O}_B -algebra structure of the relative canonical algebra $\mathcal{R} \simeq \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$ is completely described by the two morphisms σ_2 and δ . We shall use this fact later in the proof of our main theorem.

For the \mathcal{O}_B -algebras above, let us denote by $\pi_{\mathcal{C}} : \mathcal{C} = \mathcal{P}\text{roj } \mathcal{A} \rightarrow B$ and $\bar{f} : X = \mathcal{P}\text{roj } \mathcal{R} \rightarrow B$ the structure morphisms of $\mathcal{C} = \mathcal{P}\text{roj } \mathcal{A}$ and $X = \mathcal{P}\text{roj } \mathcal{R}$, respectively. Then by the two natural morphisms of \mathcal{O}_B -algebras $\mathcal{S}(V_1) \rightarrow \mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{R} \simeq \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$, we obtain the commutative diagram

$$\begin{array}{ccccc} X = \mathcal{P}\text{roj } \mathcal{R} & \xrightarrow{\psi} & \mathcal{C} = \mathcal{P}\text{roj } \mathcal{A} & \longrightarrow & \mathbb{P} = \mathcal{P}\text{roj } \mathcal{S}(V_1) \\ \downarrow \bar{f} & & \downarrow \pi_{\mathcal{C}} & & \downarrow \pi_{\mathbb{P}} \\ B & \xrightarrow{=} & B & \xrightarrow{=} & B, \end{array}$$

where ψ is the natural projection induced by $\mathcal{A} \rightarrow \mathcal{R}$ above.

Next let us describe V_2^- using the locally free sheaves V_1 and L .

Lemma 3.7. \mathcal{C} is a divisor on \mathbb{P} , and is a member of the linear system $|\mathcal{O}_{\mathbb{P}}(2) \otimes \pi_{\mathbb{P}}^*(L^{\otimes(-1)})|$.

Proof. By (1) in Lemma 3.5, we obtain the short exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}}(-2) \otimes \pi_{\mathbb{P}}^*L \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathcal{C}} \rightarrow 0.$$

From this we infer $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}}(2) \otimes \pi_{\mathbb{P}}^*(L^{\otimes(-1)})|$, hence the assertion. \square

Lemma 3.8. Put $\mathcal{M} = (\det V_1) \otimes L^{\otimes(-1)}$. Then the following hold:

- i) $\omega_{\mathcal{C}|B} \simeq \mathcal{O}_{\mathcal{C}}(-1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M})$;
- ii) $\psi_*\mathcal{O}_X \simeq \mathcal{O}_{\mathcal{C}} \oplus (\mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{C}}^*(V_2^-))$;
- iii) $\psi_*(\omega_{X|B}) \simeq (\mathcal{O}_{\mathcal{C}}(-1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M})) \oplus (\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)}))$.

Proof. The assertion i) follows from the adjunction formula together with Lemma 3.7 and $\omega_{\mathbb{P}} \simeq \mathcal{O}_{\mathbb{P}}(-3) \otimes \pi_{\mathbb{P}}^*(\omega_B \otimes \det V_1)$. The assertion ii) follows from the fact that $\psi : X = \mathcal{P}\text{roj } \mathcal{R} \rightarrow \mathcal{C} = \mathcal{P}\text{roj } \mathcal{A}$ is the morphism

induced by $\mathcal{A} \rightarrow \mathcal{R} \simeq \mathcal{A} \oplus (\mathcal{A}[-2] \otimes V_2^-)$. Thus we only need to show the assertion iii). But since $\psi : X \rightarrow \mathcal{C}$ is finite, we have $\omega_X \simeq f^! \omega_{\mathcal{C}}$, i.e., $\psi_* \omega_X \simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\psi_* \mathcal{O}_X, \omega_{\mathcal{C}})$, and hence $\psi_*(\omega_{X|B}) \simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\psi_* \mathcal{O}_X, \omega_{\mathcal{C}|B})$. Then the assertion iii) follows from i) and ii). \square

Lemma 3.9. *There exists a natural isomorphism*

$$\omega_{X|B} \rightarrow \psi^*(\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})).$$

The inverse of this isomorphism is induced by the natural inclusion

$$\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)}) \rightarrow \psi_*(\omega_{X|B}) \quad (3)$$

of the second direct summand of iii), Lemma 3.8.

Proof. As we have shown in the proof of Lemma 3.8, we have

$$\begin{aligned} \psi_*(\omega_{X|B}) &\simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\psi_* \mathcal{O}_X, \omega_{\mathcal{C}|B}) \\ &\simeq \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}, \omega_{\mathcal{C}|B}) \oplus \mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{C}}^*(V_2^-), \omega_{\mathcal{C}|B}). \end{aligned}$$

So a local section to $\psi_*(\omega_{X|B})$ is specified by a pair (s, t) , where s is a local section to $\mathcal{O}_{\mathcal{C}}(-1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M})$ identified with that to $\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}, \omega_{\mathcal{C}|B})$, and t is a local section to $\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})$ identified with that to $\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{C}}^*(V_2^-), \omega_{\mathcal{C}|B})$. In the same way, a local section to $\psi_* \mathcal{O}_X \simeq \mathcal{O}_{\mathcal{C}} \oplus (\mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{C}}^*(V_2^-))$ is specified by a pair (α, β) , where α is a local section to $\mathcal{O}_{\mathcal{C}}$, and β is a local section to $\mathcal{O}_{\mathcal{C}}(-2) \otimes \pi_{\mathcal{C}}^*(V_2^-)$.

One can easily check that via the identifications above the $\psi_* \mathcal{O}_X$ -module structure of $\psi_*(\omega_{X|B})$ is given locally by

$$(\alpha, \beta)(s, t) = (\alpha s + \beta t, \alpha t + \beta \delta s),$$

where

$$\delta \in \text{Hom}((V_2^-)^{\otimes 2}, \mathcal{A}_4) \simeq H^0(\mathcal{A}_4 \otimes (V_2^-)^{\otimes(-2)}) \simeq H^0(\mathcal{O}_{\mathcal{C}}(4) \otimes \pi_{\mathcal{C}}^*((V_2^-)^{\otimes(-2)}))$$

is the global section to $\mathcal{H}om_{\mathcal{O}_B}((V_2^-)^{\otimes 2}, \mathcal{A}_4)$ (identified with that to $\mathcal{O}_{\mathcal{C}}(4) \otimes \pi_{\mathcal{C}}^*((V_2^-)^{\otimes(-2)})$) introduced just after Proposition 3.

From this it follows that $\psi_*(\omega_{X|B})$ is locally a free $\psi_* \mathcal{O}_X$ -module of rank 1, and we can take $(0, u)$ as its local base, with u being a local base of the invertible $\mathcal{O}_{\mathcal{C}}$ -module $\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})$. If we use this to compute the natural morphism $\psi^*(\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})) \rightarrow \omega_{X|B}$ induced by (3), we see easily that this induced morphism $\psi^*(\mathcal{O}_{\mathcal{C}}(1) \otimes \pi_{\mathcal{C}}^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})) \rightarrow \omega_{X|B}$ is an isomorphism. Hence we have the assertion. \square

Corollary 3.1. *For any integer $n \geq 0$,*

$$\begin{aligned} \psi_*(\omega_{X|B}^{\otimes n}) &\simeq (\mathcal{O}_C(n) \otimes \pi_C^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes n}) \oplus \\ &\quad (\mathcal{O}_C(n-2) \otimes \pi_C^*(V_2^-) \otimes \pi_C^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes n}) \end{aligned}$$

holds. Moreover, in the above,

$$\psi_*(\omega_{X|B}^{\otimes n})^\pm \simeq \mathcal{O}_C(n) \otimes \pi_C^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes n} \quad (4)$$

$$\psi_*(\omega_{X|B}^{\otimes n})^\mp \simeq \mathcal{O}_C(n-2) \otimes \pi_C^*(V_2^-) \otimes \pi_C^*(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes n} \quad (5)$$

hold, where $\psi_(\omega_{X|B}^{\otimes n}) = \psi_*(\omega_{X|B}^{\otimes n})^+ \oplus \psi_*(\omega_{X|B}^{\otimes n})^-$ is the decomposition into eigen-sheaves with respect to the action by the hyperelliptic involution of \bar{f} .*

Proof. This follows from $\psi_*\mathcal{O}_X \simeq \mathcal{O}_C \oplus (\mathcal{O}_C(-2) \otimes \pi_C^*(V_2^-))$, Lemma 3.9, and the projection formula. \square

Proposition 4. *Let $V_2 = V_2^+ \oplus V_2^-$ be the decomposition as in the beginning of this section, and L , the kernel of $\sigma_2 : \text{Sym}^2 V_1 \rightarrow V_2^+$. Then*

$$V_2^- \simeq (\det V_1) \otimes L^{\otimes(-1)}.$$

Proof. Note that since X has at most rational double points as its singularities, we have $V_n = f_*(\omega_{S|B}^{\otimes n}) \simeq \bar{f}_*(\omega_{X|B}^{\otimes n})$ for any integer $n \geq 0$. Thus, by taking \bar{f}_* of (4) and (5) of Corollary 3.1, we obtain

$$V_1 = \bar{f}_*(\omega_{X|B}) \simeq \mathcal{A}_1 \otimes (\mathcal{M} \otimes (V_2^-)^{\otimes(-1)}) \quad (6)$$

$$V_2^- \simeq V_2^- \otimes (\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes 2} \quad (7)$$

Since the locally free sheaf $V_1 \simeq \mathcal{A}_1$ has rank 3, by taking the determinant bundles of both hands of (6), we obtain $(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes 3} \simeq \mathcal{O}_B$. And also, since V_2^- is invertible, by (7), we obtain $(\mathcal{M} \otimes (V_2^-)^{\otimes(-1)})^{\otimes 2} \simeq \mathcal{O}_B$. Thus we obtain $\mathcal{M} \otimes (V_2^-)^{\otimes(-1)} \simeq \mathcal{O}_B$. Since $\mathcal{M} = (\det V_1) \otimes L^{\otimes(-1)}$, this implies the assertion. \square

4 Proof of the main theorem

In this section, we shall prove our main theorem, i.e., Theorem 1. Let us begin with the proof of Propositions 1 and 2.

PROOF OF PROPOSITION 2

Let $f : S \rightarrow B$ be a hyperelliptic fibration of genus 3 as in Proposition 2, and $\bar{f} : X \rightarrow B$, its relative canonical model. Then X has at most

rational double points as its singularities. Thus by the same argument as in Catanese–Pignatelli [9, Theorem 4.7], $\mathcal{C} = \mathcal{P}\text{roj } \mathcal{A}$ also has at most rational double points as its singularities. From this together with Propositions 3 and 4, we see that $(B, V_1, V_2^+, \sigma_2, \delta)$ for our f is an admissible 5-tuple, hence the assertion. \square

PROOF OF PROPOSITION 1

Let $(B, V_1, V_2^+, \sigma_2, \delta)$ be an admissible 5-tuple. Construct for this 5-tuple the commutative diagram

$$\begin{array}{ccccccc} S & \xrightarrow{\pi} & X & \xrightarrow{\psi} & \mathcal{C} & \longrightarrow & \mathbb{P} \\ \downarrow f & & \downarrow \bar{f} & & \downarrow \pi_{\mathcal{C}} & & \downarrow \pi_{\mathbb{P}} \\ B & \xrightarrow{=} & B & \xrightarrow{=} & B & \xrightarrow{=} & B, \end{array}$$

following the procedure in Subsection 2.1. By repeating the computations in Section 3, we see easily that this $f : S \rightarrow B$ is a hyperelliptic fibration of genus 3 and that $\omega_{S|B} \simeq (\psi \circ \pi)^* \mathcal{O}_{\mathcal{C}}(1)$. This computation of $\omega_{S|B}$ implies that the sheaf $\omega_{S|B}|_F = \mathcal{O}_F(K_F)$ is generated by global sections for any fiber F of $f : S \rightarrow B$. So the fibration $f : S \rightarrow B$ is relatively minimal.

Thus we only need to show that all the fibers of $f : S \rightarrow B$ are 2-connected. But this follows from the following lemma:

Lemma 4.1. *Let F be a fiber of a relatively minimal hyperelliptic fibration of genus 3. Then the natural morphism*

$$\phi : \text{Sym}^2 H^0(\mathcal{O}_F(K_F)) \rightarrow H^0(\mathcal{O}_F(2K_F))^+$$

is non surjective, if and only if F fails to be 2-connected.

This lemma follows from the computation of the canonical rings of genus 3 fibers in the Thesis of Mendes Lopes [22]. As shown by Konno and Mendes Lopes in [21, Example 1] (see also [20, Theorem III]), the computation in [22] for multiple fibers includes false results. This however does not affect our lemma. To avoid to force readers to check and patch the pieces from proof in [22], we give here a proof of Lemma 4.1 in such a way that the correctness of the lemma is clear.

Proof of Lemma 4.1. We have already seen that the 2-connectedness of F implies the surjectivity of the morphism in the assertion (see the paragraph just after Lemma 3.2). Thus we only need to show the 2-connectedness of F assuming the surjectivity of the morphism. So assume contrary that the fiber F is non 2-connected. We shall show the non-surjectivity of the morphism for this case.

If F is 1-connected (but non 2-connected), then by [22, Theorem 1.18, Chapter III], the image of the multiplication map $\mathrm{Sym}^2 H^0(\mathcal{O}_F(K_F)) \rightarrow H^0(\mathcal{O}_F(2K_F))$ has codimension ≥ 2 in $H^0(\mathcal{O}_F(2K_F))$. Thus ϕ is non surjective in this case.

So assume that F is non 1-connected. Then there exists a 1-connected divisor D with $p_a(D) = 2$ such that $F = 2D$. By the standard short exact sequence

$$0 \rightarrow \mathcal{O}_D(nK_F - D) \rightarrow \mathcal{O}_F(nK_F) \rightarrow \mathcal{O}_D(nK_F) \rightarrow 0,$$

we see that $H^0(\mathcal{O}_F(nK_F)) \rightarrow H^0(\mathcal{O}_D(nK_F))$ is surjective for any $n \geq 1$ (for the proof for the case $n = 1$, use the standard fact that $\mathcal{O}_D(D)$ is a non-trivial 2-torsion in $\mathrm{Pic}^0(D)$). By the Riemann–Roch theorem, we see moreover $h^0(\mathcal{O}_D(K_F)) = 1$ and $h^0(\mathcal{O}_D(2K_F)) = 3$.

Now consider the natural commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_F(K_F))^{\otimes 2} & \longrightarrow & H^0(\mathcal{O}_D(K_F))^{\otimes 2} \\ \downarrow \lambda & & \downarrow \\ H^0(\mathcal{O}_F(2K_F)) & \xrightarrow{\mu} & H^0(\mathcal{O}_D(2K_F)), \end{array}$$

where the vertical arrows are the multiplication maps, and the horizontal arrows are the restriction maps. The image $\mathrm{Im}(\mu \circ \lambda)$ has codimension ≥ 2 in $H^0(\mathcal{O}_D(2K_F))$, since the composite $\mu \circ \lambda$ factors through $H^0(\mathcal{O}_D(K_F))^{\otimes 2} \rightarrow H^0(\mathcal{O}_D(2K_F))$ (note that we have $h^0(\mathcal{O}_D(K_F)) = 1$). Since μ is surjective, this implies that the image $\mathrm{Im} \lambda$ has codimension ≥ 2 in $H^0(\mathcal{O}_F(2K_F))$. Thus the image of ϕ has codimension ≥ 1 in $H^0(\mathcal{O}_F(2K_F))^+$, i.e., ϕ is non surjective in this case, too.

Summing these up together, we see that the fiber F needs to be 2-connected, if ϕ is surjective. \square

Now let us prove Theorem 1.

PROOF OF THEOREM 1

By the computations in Section 3, we see the existence of the one-to-one correspondence as in the assertion; in fact, given a fibration $f : S \rightarrow B$ as in the assertion, Proposition 3 and Remark 2 implies that the fibration $f : S \rightarrow B$ is completely recovered by the associated 5-tuple, and by repeating the argument in Section 3 we see that the association in Proposition 1 and that in Proposition 2 are mutually inverse.

Thus we only need to show the assertion concerning the numerical invariants of the surface S . But this follows from the following standard formula

for $\deg V_n$:

$$\chi(\mathcal{O}_S) - (3-1)(b-1) = \deg V_1 \quad (8)$$

$$c_1^2(S) - 8(3-1)(b-1) = \deg V_2 - \deg V_1, \quad (9)$$

where b is the genus of the base curve B . In fact, from the equality $\deg V_2 = \deg V_2^+ + \deg V_2^-$, the short exact sequence $0 \rightarrow L \rightarrow \text{Sym}^2 V_1 \rightarrow V_2^+ \rightarrow 0$, and the isomorphism $V_2^- \simeq (\det V_1) \otimes L^{\otimes(-1)}$, we infer

$$\deg V_2^+ = 4 \deg V_1 - \deg L \quad \deg V_2^- = \deg V_1 - \deg L,$$

hence $\deg V_2 = 5 \deg V_1 - 2 \deg L$. By this together with (8) and (9), we obtain the assertion. \square

5 Sufficient conditions for the existence of admissible 5-tuples and some examples

In this section, we shall study conditions for the existence of admissible 5-tuples for the case $B \simeq \mathbb{P}^1$, show for some cases the uniqueness of the deformation type and the simply connectedness of the resulting surfaces S 's, and give some examples of our fibrations $f : S \rightarrow B$'s including those with minimal regular S 's with $p_g = 4$ and $c_1^2 = 8$. This last examples with $p_g = 4$ and $c_1^2 = 8$ turn out to belong to the family \mathcal{M}_0 in Bauer–Pignatelli [4].

In this section, we assume $B \simeq \mathbb{P}^1$. Then by Grothendieck's theorem and Fujita's semi-positivity theorem, there exist integers $0 \leq d_0 \leq d_1 \leq d_2$ and e_0 such that $V_1 = \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$, $L = \mathcal{O}_B(e_0)$. In what follows for a relatively minimal genus 3 hyperelliptic fibration $f : S \rightarrow B = \mathbb{P}^1$ as in our main theorem, we let $\chi_f = \chi(\mathcal{O}_S) + 2$ and $K_f^2 = K_S^2 + 16$. By our main theorem, we have

$$\chi_f = \deg V_1 \quad K_f^2 = 4 \deg V_1 - 2 \deg L.$$

First note that if we denote by l and x_λ local bases of the invertible sheaves L and $\mathcal{O}_B(d_\lambda)$ ($0 \leq \lambda \leq 2$), respectively, then any \mathcal{O}_B -module homomorphism $L = \mathcal{O}_B(e_0) \rightarrow \text{Sym}^2 V_1 = \bigoplus_{i_0+i_1+i_2=2} \mathcal{O}_B(\sum_{\lambda=0}^2 i_\lambda d_\lambda)$ is given by

$$\Phi_{(\alpha_{i_0 i_1 i_2})} : l \mapsto \left(\sum_{i_0+i_1+i_2=2} \alpha_{i_0 i_1 i_2} \right) l = \sum_{i_0+i_1+i_2=2} a_{i_0 i_1 i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}, \quad (10)$$

for global sections $\alpha_{i_0 i_1 i_2} \in H^0(\mathcal{O}_B(\sum_{\lambda=0}^2 i_\lambda d_\lambda - e_0))$ (note that $\alpha_{i_0 i_1 i_2} l = a_{i_0 i_1 i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}$ is a local section to $\mathcal{O}_B(\sum_{\lambda=0}^2 i_\lambda d_\lambda)$). In our case we have $\deg V_1 = \sum_{\lambda=0}^2 d_\lambda$. So we restrict our argument in this section to the case $\sum_{\lambda=0}^2 d_\lambda > 0$, i.e., the case of fibrations with non-constant moduli.

Proposition 5. *Let $0 \leq d_0 \leq d_1 \leq d_2$ and e_0 be integers such that $\sum_{\lambda=0}^2 d_\lambda > 0$, and assume that $B \simeq \mathbb{P}^1$. If there exists an admissible 5-tuple such that $V_1 \simeq \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$ and $L \simeq \mathcal{O}_B(e_0)$, then $e_0 \leq \min\{d_0 + d_2, 2d_1\}$*

Proof. If we write our $L \rightarrow \text{Sym}^2 V_1$ of a 5-tuple as (10), the defining equation of $\mathcal{C} \in |\mathcal{O}_{\mathbb{P}}(2) \otimes \pi_{\mathbb{P}}^*(L^{\otimes(-1)})|$ is given locally by

$$\frac{\sum_{i_0+i_1+i_2=2} a_{i_0 i_1 i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}}{1} \otimes l^{\otimes(-1)} = 0.$$

Since we have $\alpha_{i_0 i_1 i_2} \in H^0(\mathcal{O}_B(\sum_{\lambda=0}^2 i_\lambda d_\lambda - e_0))$, if $d_0 + d_2 < e_0$, the global sections α_{200} , α_{110} , and α_{101} are identically zero. Then the corresponding \mathcal{C} has 1-dimensional singular locus and this 5-tuple is not admissible. So we obtain $e_0 \leq d_0 + d_2$ for an admissible 5-tuple. In the same way, we obtain $e_0 \leq 2d_1$ for an admissible 5-tuple, too. \square

Note that we have

$$3K_f^2 - 8\chi_f = 2(2(d_0 + d_2 - e_0) + (2d_1 - e_0)).$$

Thus by Proposition 5, we obtain the well known inequality $3K_f^2 - 8\chi_f \geq 0$ (but only under our restrictive assumptions).

Remark 3. By the above, if $3K_f^2 - 8\chi_f = 0$, then we have $d_0 + d_2 = e_0$ and $2d_1 = e_0$. Thus there exist integers $d > 0$ and m such that

$$V_1 = \mathcal{O}_B(d - m) \oplus \mathcal{O}_B(d) \oplus \mathcal{O}_B(d + m) \quad L = \mathcal{O}_B(2d).$$

Assume $m > 0$. Then the argument above shows that the equation of \mathcal{C} in \mathbb{P} is $(a_{101}x_0x_2 + a_{020}x_1^2 + a_{011}x_1x_2 + a_{002}x_2^2) \otimes l^{\otimes(-1)} = 0$, where a_{101} and a_{020} are nowhere vanishing functions (because $d_0 + d_2 - e_0 = 2d_1 - e_0 = 0$). In particular, the section $\Delta = \{x_1 = x_2 = 0\} \subset \mathbb{P}$ of $\pi_{\mathbb{P}} : \mathbb{P} \rightarrow B$ is contained in our relative conic \mathcal{C} . By an easy computation we see that $\mathcal{C} \rightarrow B = \mathbb{P}^1$ is the Hirzebruch–Segre surface Σ_m with minimal section $\Delta_0 = \Delta|_{\mathcal{C}}$ and with $\Delta_0^2 = -m$. Let us denote by Γ a fiber of $\mathcal{C} \rightarrow B$. Since $\omega_{\mathcal{C}} \simeq (\mathcal{O}_{\mathbb{P}}(-1) \otimes \pi_{\mathbb{P}}^*((\det V_1) \otimes L^{\otimes(-1)} \otimes \omega_B))|_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(-2\Delta_0 - (2+m)\Gamma)$, we obtain $\mathcal{O}_{\mathbb{P}}(1)|_{\mathcal{C}} \simeq \mathcal{O}_{\mathcal{C}}(2\Delta_0 + (m+d)\Gamma)$. Since the branch divisor D_0 of the projection $S \rightarrow \mathcal{C} \simeq \Sigma_m$ belongs to the linear system $|\mathcal{O}_{\mathbb{P}}(4) \otimes \pi_{\mathbb{P}}^*((V_2^-)^{\otimes(-2)})|_{\mathcal{C}}$, we see that our surface S is the minimal desingularization of the double cover of $\mathcal{C} \simeq \Sigma_m$ ($0 \leq m \leq 2d/3$) branched along a divisor $D_0 \in |8\Delta_0 + 2(2m+d)\Gamma|$ with at most negligible singularities. The final description of S is valid also for the case $m = 0$.

Remark 4. In the description of surfaces with the lowest slope in the remark above, one can easily check that if $d \leq 2$ then the surface S has Kodaira

dimension at most 1 except for the case $d = 2$ and $m = 1$. If $d = 2$ and $m = 1$, then S is not minimal, and the minimal model S^* has $K_{S^*}^2 = 2$ and $\chi(\mathcal{O}_{S^*}) = 4$, hence $K_{S^*}^2 = 2\chi(\mathcal{O}_{S^*}) - 6$. Meanwhile, if $d \geq 3$, then S is of general type. For example if $d = 3$, then we have $m \leq 2$. In this case, S is a minimal surface with $K_S^2 = 8$ and $\chi(\mathcal{O}_S) = 7$ (hence again $K_S^2 = 2\chi(\mathcal{O}_S) - 6$) except for the case $m = 2$. If $d = 3$ and $m = 2$, then S is not minimal, and the minimal model S^* has $K_{S^*}^2 = 9$ and $\chi(\mathcal{O}_{S^*}) = 7$, hence $K_{S^*}^2 = 2\chi(\mathcal{O}_{S^*}) - 5$. It is easy to check the following: if $d = 2$ and $m = 1$, S^* is in Case i), Theorem 1.6. of [14]; if $d = 3$ and $m = 0$, S is in Case iii), Theorem 1.6 of [14]; if $d = 3$ and $m = 1$, S is in Case ii), Theorem 1.6 of [14]; if $d = 3$ and $m = 2$, S^* is in Case B.2, Theorem 1.3 of [15]. In Case $d = 3$ and $m = 1$, Horikawa's description says our surface S is a minimal resolution of the double cover of \mathbb{P}^2 branched along a curve of degree 10 having at most negligible singularities. Our case of genus 3 fibration corresponds to the case where this degree 10 curve has at least one negligible singularity. Blowing up \mathbb{P}^2 at this singularity we obtain a projection from the Hirzebruch–Segre surface $\Sigma_1 \rightarrow \mathbb{P}^2$. The projection $S \rightarrow \mathbb{P}^2$ lifts to $S \rightarrow \Sigma_1 \simeq \mathcal{C}$, and our genus 3 fibration comes from the ruling $\Sigma_1 \rightarrow B = \mathbb{P}^1$ of the Hirzebruch–Segre surface. For all other cases, where the fibration comes from is immediately seen in Horikawa's description.

Next, let us study sufficient conditions for the existence of admissible 5-tuples. The following Lemma follows easily from direct computations.

Lemma 5.1. *Consider \mathcal{O}_B -module homomorphism $\tilde{\Phi}_{(\alpha_{i_0 i_1 i_2})} : L \rightarrow \text{Sym}^2 V_1$ given by*

$$l \mapsto (\alpha_{101} + \sum_{i_1+i_2=2} \alpha_{0i_1 i_2}) l = (a_{101}x_0x_2 + \sum_{i_1+i_2=2} a_{0i_1 i_2}x_1^{i_1}x_2^{i_2}),$$

and let $\mathcal{C} \subset \mathbb{P} = \mathbb{P}(V_1)$ be the relative conic associated to this $L \rightarrow \text{Sym}^2 V_1$. Let $p \in B$. Then we have the following:

- 1) If $\alpha_{101}(p) \neq 0$ and $\alpha_{020}(p) \neq 0$, then the fiber over p of $\mathcal{C} \rightarrow B = \mathbb{P}^1$ is non-singular.
- 2) Assume $\alpha_{020}(p) = 0$. If $\alpha_{101}(p) \neq 0$, then the only possible singularity of \mathcal{C} lying over p is $(p, (x_0 : x_1 : x_2)) = (p, (-a_{011}(p) : a_{101}(p) : 0))$. This point is a singularity of \mathcal{C} if and only if $n := \text{ord}_p \alpha_{020} - 1 \geq 1$. In this last case it is a singularity of type A_n of \mathcal{C} .
- 3) Assume $\alpha_{101}(p) = 0$. If $\alpha_{002}(p) \neq 0$ and $(4\alpha_{020}\alpha_{002} - (\alpha_{011})^2)(p) \neq 0$ are satisfied, and neither α_{101} nor α_{020} is identically zero, then the unique singularity of \mathcal{C} lying over $p \in B$ is $(p, (x_0 : x_1 : x_2)) = (p, (1 : 0 : 0))$. This point is a singularity of type A_n of \mathcal{C} , where $n := 2\text{ord}_p \alpha_{101} + \text{ord}_p \alpha_{020} - 1$ (≥ 1).

Let us use this lemma to prove the following:

Proposition 6. *Let $B = \mathbb{P}^1$, and let $0 \leq d_0 \leq d_1 \leq d_2$ and e_0 be integers such that $\sum_{\lambda=0}^2 d_\lambda > 0$. Assume either*

- A) $\sum_{\lambda=0}^2 d_\lambda - 2d_0 \leq e_0 \leq \min\{d_0 + d_2, 2d_1\}$, or
- B) $\sum_{\lambda=0}^2 d_\lambda - \varepsilon/2 \leq e_0 \leq 2d_0$, where $\varepsilon := \min\{d_0 + 3d_2, 3d_1 + d_2\}$.

Then there exists an admissible 5-tuple (for relatively minimal 2-connected genus 3 hyperelliptic fibrations) such that $V_1 \simeq \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$ and $L \simeq \mathcal{O}_B(e_0)$, where $0 \rightarrow L \rightarrow \text{Sym}^2 V_1 \rightarrow V_2^+ \rightarrow 0$ is the short exact sequence in the structure theorem. In case A) an admissible 5-tuple can be taken in such a way that the branch divisor of $S \rightarrow \mathcal{C}$ is non-singular. In case B) an admissible 5-tuple can be taken in such a way that the variety \mathcal{C} is non-singular.

Proof. Let us first prove the assertion for Case A). Let d_0, d_1, d_2 , and e_0 be integers as in Case A) of the assertion, and put $V_1 = \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$ and $L = \mathcal{O}_B(e_0)$. For each collection $\alpha_{101}, \{\alpha_{0i_1i_2}\}$ ($i_1 + i_2 = 2, i_1, i_2 \geq 0$), let us consider the morphism $\tilde{\Phi}_{(\alpha_{101}, \{\alpha_{0i_1i_2}\})} : L \rightarrow \text{Sym}^2 V_1$ given in Lemma 5.1. Then under the conditions in A), it is an immediate consequence of Lemma 5.1 that for general α_{101} and $\{\alpha_{0i_1i_2}\}$ the \mathcal{O}_B -module $V_2^+ = \text{Cok}(\tilde{\Phi}_{(\alpha_{101}, \{\alpha_{0i_1i_2}\})} : L \rightarrow \text{Sym}^2 V_1)$ is locally free, and that the relative conic $\mathcal{C} \subset \mathbb{P}$ determined by $\tilde{\Phi}_{(\alpha_{101}, \{\alpha_{0i_1i_2}\})}$ has at most rational double points as its singularities (chose $\alpha_{101}, \alpha_{020}, \alpha_{002}$, and α_{011} in this order). Thus we only need to show the existence of $(V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ that induces an associated canonical model X with at most rational double points as its singularities, where $\mathcal{A}_4 = \text{Cok}(L \otimes \text{Sym}^2 V_1 \rightarrow \text{Sym}^4 V_1)$ is the invertible sheaf as in the definition of a 5-tuple. For this purpose, it is enough to show that for a general $(V_2^-)^{\otimes 2} \rightarrow \text{Sym}^4 V_1$ the induced branch divisor $\mathcal{D}|_{\mathcal{C}}$ of $S \rightarrow \mathcal{C}$ is non-singular, where $\mathcal{D} \in |\mathcal{O}_{\mathbb{P}}(4) \otimes \pi_{\mathbb{P}}^*(V_2^-)^{\otimes(-2)}|$ is the divisor of \mathbb{P} determined by the morphism $(V_2^-)^{\otimes 2} \rightarrow \text{Sym}^4 V_1 \simeq \pi_{\mathbb{P}*} \mathcal{O}_{\mathbb{P}}(4)$ above.

Note that we have $\text{Sym}^4 V_1 \simeq \bigoplus_{j_0+j_1+j_2=4} \mathcal{O}_B(\sum_{\lambda=0}^2 j_\lambda d_\lambda)$ and $(V_2^-)^{\otimes 2} \simeq \mathcal{O}_B(2(\sum_{\lambda=0}^2 d_\lambda - e_0))$. Thus any morphism $(V_2^-)^{\otimes 2} \rightarrow \text{Sym}^4 V_1$ is given as

$$\Psi_{(\beta_{j_0j_1j_2})} : y^{\otimes 2} \mapsto \left(\sum_{j_0+j_1+j_2=4} \beta_{j_0j_1j_2} y^{\otimes 2} = \sum_{j_0+j_1+j_2=4} b_{j_0j_1j_2} x_0^{j_0} x_1^{j_1} x_2^{j_2} \right)$$

for $\beta_{j_0j_1j_2} \in H^0(\mathcal{O}_B(\sum_{\lambda=0}^2 j_\lambda d_\lambda) \otimes (V_2^-)^{\otimes(-2)})$ ($j_0 + j_1 + j_2 = 4, j_0, j_1, j_2 \geq 0$), where y is a local base of V_2^- . Via the composition with the natural morphism $\text{Sym}^4 V_1 \rightarrow \pi_{\mathbb{P}*} \mathcal{O}_{\mathbb{P}}(4)$, the morphism $\Psi_{(\beta_{j_0j_1j_2})}$ determines an element in $H^0(\mathcal{O}_{\mathbb{P}}(4) \otimes \pi_{\mathbb{P}}^*(V_2^-)^{\otimes(-2)})$. We define the morphism

$$\Psi : \bigoplus_{j_0+j_1+j_2=4} H^0(\mathcal{O}_B(\sum_{\lambda=0}^2 j_\lambda d_\lambda) \otimes (V_2^-)^{\otimes(-2)}) \rightarrow H^0(\mathcal{O}_{\mathbb{P}}(4) \otimes \pi_{\mathbb{P}}^*(V_2^-)^{\otimes(-2)})$$

by $(\beta_{j_0 j_1 j_2}) \mapsto \Psi_{(\beta_{j_0 j_1 j_2})}$.

By the condition $\sum_{\lambda=0}^2 d_\lambda - 2d_0 \leq e_0$, we see that the linear system $|\mathcal{O}_B(\sum_{\lambda=0}^2 j_\lambda d_\lambda) \otimes (V_2^-)^{\otimes(-2)}|$ is free from base points for any $j_0, j_1, j_2 \geq 0$ such that $\sum_{\lambda=0}^2 j_\lambda = 4$. Thus the image $\text{Im } \Psi$ determines a linear system free from base points. It follows that, for a general member \mathcal{D} of $|\text{Im } \Psi|$, the restriction $D_0 = \mathcal{D}|_{\mathcal{C}}$ to \mathcal{C} is non-singular. Thus $\Psi_{(\beta_{j_0 j_1 j_2})} : (V_2^-)^{\otimes 2} \rightarrow \text{Sym}^4 V_1$ ($\rightarrow \mathcal{A}_4$) induces an admissible 5-tuple, hence the assertion for Case A).

The assertion for Case B) can be proved by the same method, but with a bit more computation. Here we only observe the difference from the proof for Case A), and give an outline of the proof to the extent that the readers can verify the detail of the proof by themselves.

First, note that the variety $\mathbb{P} = \mathbb{P}(V_1)$ contains two subvarieties $\Delta = \{x_1 = x_2 = 0\} \subset \mathbb{P}$ and $\mathcal{D}'_1 = \{x_2 = 0\} \simeq \mathbb{P}(\mathcal{O}_B(d_0) \oplus \mathcal{O}_B(d_1)) \subset \mathbb{P}$. The variety Δ is a section of the projection $\pi_{\mathbb{P}} : \mathbb{P} \rightarrow B = \mathbb{P}^1$, and the variety $\mathcal{D}'_1 = \{x_2 = 0\}$ is a divisor of \mathbb{P} and a member of the linear system $|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \mathcal{O}_B(-d_2)|$. In case B), instead of $\tilde{\Phi}_{(\alpha_{i_0 i_1 i_2})}$, we employ the morphism $\Phi_{(\alpha_{i_0 i_1 i_2})} : L \rightarrow \text{Sym}^2 V_1$ given in (10). We define the linear map $\Phi : \bigoplus_{i_0+i_1+i_2=2} H^0(\mathcal{O}_B(\sum_{\lambda=0}^2 i_\lambda d_\lambda - e_0)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}}(2) \otimes \pi_{\mathbb{P}}^* L^{\otimes(-1)})$ by $(\alpha_{i_0 i_1 i_2}) \mapsto \Phi_{(\alpha_{i_0 i_1 i_2})}$. Then by the condition $e_0 \leq 2d_0$ we see that the image $\text{Im } \Phi$ determines a base point free linear system. Thus for a general $(\alpha_{i_0 i_1 i_2})$ the induced relative conic $\mathcal{C} \subset \mathbb{P}$ is non-singular, does not contain Δ , and intersects transversally at (if any) each point of $\mathcal{C} \cap \Delta$.

Meanwhile, the control of $(\beta_{j_0 j_1 j_2})$ becomes a bit more complicated compared to that in Case A). In Case B), at the worst case of (d_0, d_1, d_2) and e_0 , the seven sections $\beta_{j_0 j_1 0}$ ($j_0 + j_1 = 4$), $\beta_{3 0 1}$, and $\beta_{2 0 2}$ vanish. So, in Case B), in stead of $\Psi_{(\beta_{j_0 j_1 j_2})}$, we employ the morphism $\tilde{\Psi}_{(\beta_{j_0 j_1 j_2})} : (V_2^-)^{\otimes 2} \rightarrow \text{Sym}^4 V_1$

$$y^{\otimes 2} \mapsto (\beta_{1 0 3} + \sum_{j_1+j_2=4, j_2 \geq 1} \beta_{0 j_1 j_2}) y^{\otimes 2} = b_{1 0 3} x_0 x_2^3 + \sum_{j_1+j_2=4, j_2 \geq 1} b_{0 j_1 j_2} x_1^{j_1} x_2^{j_2}$$

associated to $\beta_{1 0 3} \in H^0(\mathcal{O}_B(d_0+3d_2) \otimes (V_2^-)^{\otimes(-2)})$ and $\beta_{0 j_1 j_2} \in H^0(\mathcal{O}_B(j_1 d_1 + j_2 d_2) \otimes (V_2^-)^{\otimes(-2)})$ ($j_1 + j_2 = 4$, $j_1 \geq 0$, $j_2 \geq 1$), where y is a local base of V_2^- as usual.

Since the equation of the divisor $\mathcal{D} \in |\mathcal{O}_{\mathbb{P}}(4) \otimes \pi_{\mathbb{P}}^* (V_2^-)^{\otimes(-2)}|$ associated to $\tilde{\Psi}_{(\beta_{j_0 j_1 j_2})}$ has x_2 as a prime factor, we obtain the splitting $\mathcal{D} = \mathcal{D}'_1 + \mathcal{D}'_2$, where $\mathcal{D}'_1 = \{x_2 = 0\} \in |\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \mathcal{O}_B(-d_2)|$ and $\mathcal{D}'_2 \in |\mathcal{O}_{\mathbb{P}}(3) \otimes \pi_{\mathbb{P}}^* (\mathcal{O}_B(d_2) \otimes (V_2^-)^{\otimes(-2)})|$. So let us denote by

$$\Psi'_{(\beta_{j_0 j_1 j_2})} : (V_2^-)^{\otimes 2} \rightarrow (\text{Sym}^3 V_1) \otimes \mathcal{O}_B(d_2) \simeq \pi_{\mathbb{P}*} (\mathcal{O}_{\mathbb{P}}(3) \otimes \pi_{\mathbb{P}}^* \mathcal{O}_B(d_2))$$

the morphism corresponding to the defining equation in \mathbb{P} of the divisor

\mathcal{D}'_2 , and consider the linear map $\Psi' : (\beta_{j_0 j_1 j_2}) \mapsto \Psi'_{(\beta_{j_0 j_1 j_2})} \in H^0(\mathcal{O}_{\mathbb{P}}(3) \otimes \pi_{\mathbb{P}}^*(\mathcal{O}_B(d_2) \otimes (V_2^-)^{\otimes(-2)}))$.

From the condition $\sum_{\lambda=0}^2 d_{\lambda} - \varepsilon/2 \leq e_0$, we see the following:

a) The base locus of the linear system $|\mathrm{Im} \Psi'|$ determined by the image $\mathrm{Im} \Psi'$ is contained in the subvariety $\Delta = \{x_1 = x_2 = 0\} \subset \mathbb{P}$. So for a general \mathcal{D}'_2 , its restriction $\mathcal{D}'_2|_{\mathcal{C}}$ to \mathcal{C} is non-singular outside $\Delta \cap \mathcal{C}$.

b) If we take general \mathcal{C} , then the divisor $\mathcal{D}'_1|_{\mathcal{C}}$ of \mathcal{C} is non-singular.

c) If we take \mathcal{C} sufficiently general, then for a general \mathcal{D}'_2 , the divisor $(\mathcal{D}'_1 + \mathcal{D}'_2)|_{\mathcal{C}}$ of \mathcal{C} has at each point (if any) of $\Delta \cap \mathcal{C}$ a negligible singularity. More precisely, these are singularities of type $(x_2/x_0)((x_2/x_0)^2 + (x_1/x_0)^3) = 0$, i.e., those corresponding to singularities of type E_7 of the double cover (note here that the condition $\sum_{\lambda=0}^2 d_{\lambda} - \varepsilon/2 \leq e_0$ ensures non-vanishing of general β_{103} and β_{031} , which are coefficients of $(x_2/x_0)^3$ and $(x_1/x_0)^3(x_2/x_0)$, respectively, and that (x_1/x_0) and (x_2/x_0) form a system of local coordinates of \mathcal{C} around these points).

d) At points outside $\Delta \cap \mathcal{C}$, general $\mathcal{D}'_1|_{\mathcal{C}}$ and $\mathcal{D}'_2|_{\mathcal{C}}$ at most intersect each other transversally (by the defining equation of \mathcal{D}_2 , we see that, for a general \mathcal{C} , intersection points of $\mathcal{D}'_1|_{\mathcal{C}}$ and $\mathcal{D}'_2|_{\mathcal{C}}$ outside $\Delta \cap \mathcal{C}$ appear only in the fibers over points $p \in B$'s such that $\beta_{031}(p) = 0$).

Thus, for a general $(\beta_{j_0 j_1 j_2})$, the divisor $\mathcal{D}|_{\mathcal{C}} = (\mathcal{D}'_1 + \mathcal{D}'_2)|_{\mathcal{C}}$ of \mathcal{C} has at most negligible singularities. This implies that $(V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ associated to a general $(\beta_{j_0 j_1 j_2})$ induces an admissible 5-tuple, hence the assertion for Case B). \square

Remark 5. Note that the cases of the lowest slope $8/3$ are covered by Case A) of Proposition 6 (see Remark 3). As is well-known, contrary to the case of genus 2 fibrations, the slope of a genus 3 fibration can take higher values even if it has only 2-connected fibers. Using our Proposition 6, we can for example show that the slope can take the value s for any rational number $8/3 \leq s \leq 10/3$ even under the 2-connected assumption. In fact, for a sufficiently large positive integer d such that $sd/2$ is an integer, put $d_0 = d_1 = d_2 = d$ and $e_0 = (4 - s)3d/2$. Then by A) of Proposition 6 we see the existence of an admissible 5-tuple such that $V_1 = \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_{\lambda})$ and $L = \mathcal{O}_B(e_0)$, which yields a 2-connected genus 3 fibration with slope s . To construct examples with even higher slope, the condition in B) is more useful. Put for example $d_0 = d$, $d_1 = 5d$, $d_2 = 7d$, and $e_0 = 2d$ for any positive integer d . Then the existence of an admissible 5-tuple is assured by case B) of Proposition 6, and the resulting fibration has slope $48/13 = 3.6923 \dots$

Next, let us show for some cases the uniqueness of the deformation type and the simply connectedness of the resulting surfaces S 's. As for the following Theorem 2, we can show with messy computations that the maximal slope

covered in this theorem is $10/3$, hence < 3.5 . So the simply connectedness follows from Xiao's results [30, Lemma 2, Theorem 2], since our fibrations $f : S \rightarrow B$'s have no multiple fiber. In Proposition 7 (see also Remark 6), however, we shall give examples with slope 3.6. In this last case, the simply connectedness does not directly follow from [30, Lemma 2, Theorem 2]. To unify the the proof, we shall give even in Theorem 2 a proof of simply connectedness using a result from Catanese [5], which works also for the cases in Proposition 7.

Theorem 2. *Let $f : S \rightarrow B = \mathbb{P}^1$ be a relatively minimal hyperelliptic fibration of genus 3 with all fibers 2-connected. Assume $f_*(\omega_{S|B}) = V_1 \simeq \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$ with $\sum_{\lambda=0}^2 d_\lambda > 0$ and $d_0 \leq d_1 \leq d_2$. Let $L = \ker \sigma_2$ be the kernel of the morphism $\sigma_2 : \text{Sym}^2 V_1 \rightarrow V_2^+$ as in Theorem 1. Then if $\sum_{\lambda=0}^2 d_\lambda - 2d_0 \leq e_0 \leq \min\{d_0 + d_2, 2d_1\}$, then the surface S is topologically simply connected, and any two such S 's having the same (d_0, d_1, d_2) and e_0 are equivalent under the deformation of complex structures.*

Proof. Let us first prove the uniqueness of the deformation type. Assume that (d_0, d_1, d_2) and e_0 as in the assertion are given. Fix one $0 \rightarrow L \rightarrow \text{Sym}^2 V_1 \rightarrow V_2^+ \rightarrow 0$ (exact) such that the associated relative conic $\mathcal{C} \subset \mathbb{P}$ has at most rational double points. Under this fixed \mathcal{C} , deforming the morphism $(V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ correspond to deforming the branch divisor of $S \rightarrow \mathcal{C}$. Thus with the aid of Tjurina's theorem on simultaneous resolution of the family of rational double points we see that under the fixed \mathcal{C} the deformation type of the resulting surface S does not depend on the choice of admissible $(V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ (note here that to a general member D of the linear system to which the branch divisors of $S \rightarrow \mathcal{C}$ belong, an admissible $(V_2^-)^{\otimes 2} \rightarrow \mathcal{A}_4$ correspond, since in our case D has at most negligible singularities). We denote by $d(\mathcal{C})$ this deformation type, which depends only on the choice of the relative conic \mathcal{C} .

Since $\sum_{\lambda=0}^2 d_\lambda - 2d_0 \leq e_0$, the linear system determined by the image $\text{Im} \Psi$ of Ψ (in the proof of Proposition 6) is base point free. Thus there exists a member $\mathcal{D} \in |\text{Im} \Psi|$ whose restriction $\mathcal{D}|_{\mathcal{C}}$ is non-singular and passes no singular point of \mathcal{C} . Under this fixed \mathcal{D} , if \mathcal{C}' is any sufficiently small deformation of \mathcal{C} , the deformation type of the surface associated to the pair $(\mathcal{C}', \mathcal{D})$ is the same as that of the surface associated to the original pair $(\mathcal{C}, \mathcal{D})$. This means $d(\mathcal{C}) = d(\mathcal{C}')$. Since the space parametrizing \mathcal{C} 's are connected, this implies the uniqueness of the deformation type of our S .

Now let us prove the simply connectedness of the surface S . Since we have already proved the uniqueness of the deformation type of our S , we only need to find an S that is simply connected. To do this, let us take \mathcal{D} as

above such that the restriction $D_0 = \mathcal{D}|_{\mathcal{C}}$ is non-singular, and show that the associated surface S is simply connected.

Let $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ be the minimal desingularization of \mathcal{C} , and \tilde{D}_0 , the total transform of D_0 to $\tilde{\mathcal{C}}$. Note that the linear system $|\tilde{D}_0|$ is free from base point (because $\sum_{\lambda=0}^2 d_\lambda - 2d_0 \leq e_0$). From this together with $\tilde{D}_0^2 = 16e_0 > 0$, we see that the divisor \tilde{D}_0 is flexible in the sense of Definition 1.4., Catanese [5]. Moreover, our variety $\tilde{\mathcal{C}}$ is rational, hence simply connected. Then the simply connectedness of S follows from Proposition 1.8., Catanese [5]. \square

By a similar argument we can prove the following:

Proposition 7. *Let $0 \leq d_0 \leq d_1 \leq d_2$ and e_0 be integers such that $\sum_{\lambda=0}^2 d_\lambda > 0$. Assume either*

- A) $\sum_{\lambda=0}^2 d_\lambda - (3d_0 + d_1)/2 \leq e_0 \leq 2d_0$, or
- B) $\sum_{\lambda=0}^2 d_\lambda - \varepsilon'/2 \leq e_0 = 2d_0$, where $\varepsilon' = \min\{d_0 + 3d_2, 4d_1\}$.

Then there exists a relatively minimal genus 3 hyperelliptic fibration $f : S \rightarrow B = \mathbb{P}^1$ with all fibers 2-connected such that $V_1 \simeq \bigoplus_{\lambda=0}^2 \mathcal{O}_B(d_\lambda)$ and $L \simeq \mathcal{O}_B(e_0)$ with S topologically simply connected.

Prof. Since $3d_0 + d_1 \leq \varepsilon' \leq \varepsilon$, where ε is as in Proposition 6, the existence of a relatively minimal fibration $f : S \rightarrow B$ with all fibers 2-connected is assured by Proposition 6. Thus we only need to prove that we can take $f : S \rightarrow B$ such that S is simply connected.

Before we start, let us note that in both Cases A) and B) we have $e_0 > 0$. In fact, in Case A), we have $e_0 \geq \sum_{\lambda=0}^2 d_\lambda - (3d_0 + d_1)/2 = (d_1 - d_0)/2 + d_2 \geq 0$, and in Case B) we have $2d_0 = e_0 \geq \sum_{\lambda=0}^2 d_\lambda - \varepsilon'/2 \geq \sum_{\lambda=0}^2 d_\lambda - ((d_0 + 3d_2) + 4d_1)/4 = (3d_0 + d_2)/4 \geq 0$. Thus, in both cases, $e_0 \leq 0$ would imply $d_0 = d_1 = d_2 = 0$, which contradicts our assumption $\sum_{\lambda=0}^2 d_\lambda > 0$.

Let us prove the assertion for Case A). The condition $e_0 \leq 2d_0$ ensures that a general $\mathcal{C} \subset \mathbb{P}$ is non-singular, and \mathcal{C} intersects $\Delta = \{x_1 = x_2 = 0\}$ transversally at (if any) each point of $\mathcal{C} \cap \Delta$ (because $\alpha_{200} \in H^0(\mathcal{O}_B(2d_0 - e_0))$, $2d_0 - e_0 \geq 0$). Then recall that our \mathcal{D} is locally defined by

$$\left(\sum_{j_0+j_1+j_2=4} \beta_{j_0 j_1 j_2} \right) y^{\otimes 2} = \sum_{j_0+j_1+j_2=4} b_{j_0 j_1 j_2} x_0^{j_0} x_1^{j_1} x_2^{j_2} = 0$$

as in the proof of Proposition 6. So the condition $\sum_{\lambda=0}^2 d_\lambda - (3d_0 + d_1)/2 \leq e_0$ implies that the image of Ψ in the proof for case A) of Proposition 6 gives a linear system with base locus contained in Δ (because of the non-vanishing of general β_{040} and β_{004}), that for a general member \mathcal{D} of this linear system its restriction $D_0 = \mathcal{D}|_{\mathcal{C}}$ is smooth at these base points (because of the non-vanishing of general β_{301}), and that for any two general \mathcal{D} 's their restrictions

$D_0 = \mathcal{D}|_{\mathcal{C}}$'s intersect each other transversally at these base points (because of the non-vanishing of general β_{310} and β_{301}). These together with $D_0^2 = 16e_0 > 0$ ensure that a general $D_0 = \mathcal{D}|_{\mathcal{C}}$ is flexible, hence the assertion for Case A).

Let us prove the assertion for case B). By the condition $e_0 = 2d_0$, a general \mathcal{C} is non-singular and satisfies $\mathcal{C} \cap \Delta = \emptyset$ (because $\alpha_{200} \in H^0(\mathcal{O}_B(2d_0 - e_0))$, $2d_0 - e_0 = 0$). But, from the condition $\sum_{\lambda=0}^2 d_\lambda - \varepsilon'/2 \leq e_0$, we see again that the image $\text{Im } \Psi$ determines a linear system with base locus contained in Δ . So the linear system $\{D_0 = \mathcal{D}|_{\mathcal{C}}\}$ is free from base points. This together with $D_0^2 = 16e_0 > 0$ ensures that a general D_0 is flexible, hence the assertion for Case B). \square

Remark 6. As an example, consider the case $d_0 = d$, $d_1 = 4d$, $d_2 = 5d$, and $e_0 = 2d$ for a positive integer $d > 0$. This case is covered by Case B) of Proposition 7. Thus we obtain an example of $f : S \rightarrow B \simeq \mathbb{P}^1$ with $V_1 \simeq \mathcal{O}_B(d) \oplus \mathcal{O}_B(4d) \oplus \mathcal{O}_B(5d)$, $L \simeq \mathcal{O}_B(2d)$, and topologically simply connected S . This S has numerical invariants $c_1^2 = 36d - 16$ and $\chi(\mathcal{O}_S) = 10d - 2$, hence the slope of f being $36/10 = 3.6$, and the ratio $c_1^2/\chi(\mathcal{O}_S)$ asymptotically being 3.6. Note even for the case $c_1^2 < 3\chi(\mathcal{O}_S)$, even the algebraic fundamental group is not necessarily trivial (see for example Mendes Lopes–Pardini [23]).

Remark 7. In Proposition 7, we put the conditions in order to assure that the general divisor D_0 is smooth at points in $\Delta \cap \mathcal{C}$. It is obvious that allowing some mild negligible singularities at points in $\Delta \cap \mathcal{C}$, we can weaken the conditions in Proposition 7. We do not execute it here, and instead chose to execute it when we need, since the conditions in the results will be a bit more complicated.

One of the interesting cases that are covered by our Proposition 7 is the case $d_0 = 1$, $d_1 = d_2 = 3$, and $e_0 = 2$. In this case, for a general admissible 5-tuple, the associated hyperelliptic fibration $f : S \rightarrow B = \mathbb{P}^1$ has minimal regular S with $c_1^2 = 8$ and $p_g = 4$. In our case, obviously, the canonical map of our S factors through the hyperelliptic involution ι of the fibration $f : S \rightarrow B$. Thus these S are of one of the types given in the list by I. Bauer and R. Pignatelli [4] of minimal regular surfaces with $c_1^2 = 8$ and $p_g = 4$ with canonical involution.

For the use in a sequel to this paper, let us specify of which type in the list in [4] our surfaces S 's are.

Proposition 8. *Let $d_0 = 1$, $d_1 = d_2 = 3$, and $e_0 = 2$, hence a special case of Case B) of Proposition 7. Let $f : S \rightarrow B = \mathbb{P}^1$ be a relatively minimal genus 3 hyperelliptic fibration with all fibers 2-connected obtained by Proposition 7 using a non-singular \mathcal{C} such that $\mathcal{C} \cap \Delta = \emptyset$. Then S is a minimal regular*

surface with $c_1^2 = 8$ and $p_g = 4$ whose corresponding point $[S]$ in the moduli space lies in the strata \mathcal{M}_0 in the Main Theorem of Bauer–Pignatelli [4].

Proof. The minimality of S follows from $\mathcal{C} \cap \Delta = \emptyset$ and $f_*(\omega_S) \simeq V_1 \otimes \omega_B \simeq \mathcal{O}_B(-1) \oplus \mathcal{O}_B(1) \oplus \mathcal{O}_B(1)$, since these imply that the canonical system of S is free from base points. The equalities $c_1^2 = 8$, $p_g = 4$, and $q = 0$ follow from Theorem 1 and $p_g = h^0(V_1 \otimes \omega_B)$. Thus we only need to show that the corresponding point $[S]$ in the moduli space lies in the strata \mathcal{M}_0 .

To prove the assertion concerning the corresponding point $[S]$, let us first note that, in Proposition 7, if $\tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is the minimal resolution of singularities of \mathcal{C} , there exist a Hirzebruch–Segre surface $\tilde{\mathcal{C}} \rightarrow B$ and a projection $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$ compatible with the projections to B such that this $\tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}$ is a blowing-up of (possibly infinite near) $(2 \sum_{\lambda=0}^2 d_\lambda - 3e_0)$ points. In fact, this follows from $K_{\tilde{\mathcal{C}}}^2 = 8 - (2 \sum_{\lambda=0}^2 d_\lambda - 3e_0)$, which in turn follows from $(\mathcal{O}_{\mathbb{P}}(1)|_{\mathcal{C}})^2 = 2 \sum_{\lambda=0}^2 d_\lambda - e_0$. In our case of $d_0 = 1$, $d_1 = d_2 = 3$, and $e_0 = 2$ with non-singular \mathcal{C} , we have $\tilde{\mathcal{C}} = \mathcal{C}$ and the projection $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$ is a blowing-up of 8 points. In particular, we obtain $K_{\mathcal{C}}^2 = 0$ for our case.

Let ι be the hyperelliptic involution of $f : S \rightarrow B$. Then the canonical map of S factors through ι , and the involution ι has no isolated fixed point because the branch divisor $D_0 = \mathcal{D}|_{\mathcal{C}}$ has at most negligible singularities. Thus our S corresponds to a point in the strata $\mathcal{M}_0^{\text{div}}$ or to a point in the strata \mathcal{M}_0 (see Section 3 of [4], especially, Proposition 3.2, Theorem 3.3, and Theorem 3.5).

Let us show that the corresponding point $[S]$ belongs to the strata \mathcal{M}_0 . To do this, since we already have $K_{\mathcal{C}}^2 = 0$, we only need to show that \mathcal{C} has no (-1) -curve contained in the image by $S \rightarrow \mathcal{C}$ of a fundamental cycle of S (see Proposition 3.2, Theorem 3.5, [4]).

Recall we have $V_1 \simeq \mathcal{O}_B(1) \oplus \mathcal{O}_B(3) \oplus \mathcal{O}_B(3)$, hence $f_*(\omega_S) \simeq \mathcal{O}_B(-1) \oplus \mathcal{O}_B(1) \oplus \mathcal{O}_B(1)$. From this we infer $H^0(\omega_S) \simeq H^0(\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \omega_B) \simeq \mathbb{C}^4$. Thus the canonical map of S is the composition of the following three maps: the natural projection $S \rightarrow \mathcal{C}$, the natural inclusion $\mathcal{C} \rightarrow \mathbb{P} = \mathbb{P}(V_1)$, and the rational map $\Phi_{|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \omega_B|} : \mathbb{P} \dashrightarrow \mathbb{P}^3$ associated to the linear system $|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \omega_B|$. But it is easy to see the following:

- a) the indeterminacy locus of $\Phi_{|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \omega_B|}$ is $\Delta = \{x_1 = x_2 = 0\}$;
- b) the image of $\Phi_{|\mathcal{O}_{\mathbb{P}}(1) \otimes \pi_{\mathbb{P}}^* \omega_B|}$ is a non-singular quadric $\mathcal{Q} \simeq B \times \mathbb{P}^1 \subset \mathbb{P}^3$ (with $\mathbb{P} \dashrightarrow \mathcal{Q}$ compatible with the natural projections onto B of \mathbb{P} and $\mathcal{Q} \simeq B \times \mathbb{P}^1$);
- c) for any $p \in B$, the restriction $\pi_{\mathbb{P}}^{-1}(p) \simeq \mathbb{P}^2 \dashrightarrow \{p\} \times \mathbb{P}^1 (\subset B \times \mathbb{P}^1)$ (to the fibers over p) of the rational map $\mathbb{P} \dashrightarrow \mathcal{Q}$ is the linear projection of $\pi_{\mathbb{P}}^{-1}(p) \simeq \mathbb{P}^2$ from the center $\pi_{\mathbb{P}}^{-1}(p) \cap \Delta$.

From these together with $\mathcal{C} \cap \Delta = \emptyset$, it follows that $\mathcal{C} \rightarrow \mathcal{Q} \subset \mathbb{P}^3$ contracts no curve, and is of mapping degree 2. Thus the canonical map Φ_{K_S} of S contracts no (-2) -curve except those contracted by $S \rightarrow \mathcal{C}$. Thus \mathcal{C} has no (-1) -curve contained in the image of a fundamental cycle of S . \square

Note that the surfaces S 's in the Proposition above are topologically simply connected. For regular surfaces with $c_1^2 = 8$ and $p_g = 4$ with non trivial torsion, see [11]. In a sequel to this paper, the example in Proposition 8 will be further studied, being deformed to surfaces with non-hyperelliptic genus 3 fibrations.

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